# On the Use of Integral Geometry for Interference Modeling and Analysis in Wireless Networks

V. Petrov, *Student Member, IEEE*, D. Moltchanov, P. Kustarev, J. M. Jornet, *Member, IEEE*, and Y. Koucheryavy, *Senior Member, IEEE*

***Abstract*— Stochastic geometry has recently gained popularity for performance assessment of wireless networks. The application of this approach is conditioned on the availability of the closed-form expressions for probability distributions of the distance between the receiver and interfering nodes. These expressions are available for the simple point processes, e.g., the Poisson point process, and area geometries only. The motivation of this letter is to draw the attention of the wireless research community to a new methodology for estimating interference in spatially random networks. We show that using the integral geometry and working with *interference sets* rather than *interference distances* one can approximate the interference for any convex deployment areas and general distribution of interfering nodes.**

***Index Terms*— Interference, integral geometry, mobile wireless networks, general node distribution, arbitrary deployments.**

## I. Introduction

THE wireless networks densification, novel networking mechanisms such as device-to-device (D2D) communications as well as the trend to use multiple access technologies to serve the mobile users all have led to the increased randomness of the network, where the positions of servicing stations such as base stations (BS), relays and D2D partners tend to be random [1]. As an attempt to incorporate probabilistic features into the performance evaluation of modern and future mobile wireless systems stochastic geometry framework has been developed [2]. Pioneered by the seminal study of Bacelli and Zuev [3], and extended by the groups of M. Haenggi, R. Heath, J. Andrews, E. Hossian, among others, researchers have recently provided notable contributions to understanding of the performance trade-offs in wireless networks.

The starting point in stochastic geometry analysis is a spatial point process modeling the nodes' locations. While there are a number of models of such processes suitable for different deployments, only few of them allow for rigorous analysis of systems' performance, e.g., the Poisson point process (PPP). The reason for analytical tractability of PPP is that the points of PPP falling into a closed set $A \subset \Re^2$ are uniformly distributed over the area of $A$. For symmetric geometries of $A$ this property allows to obtain the distribution of the distance between randomly selected points and proceed with interference analysis [2]. Unfortunately, this approach is no longer feasible for more complex point processes and/or typical indoor and outdoor locations, such as a shopping mall, a square or an auditorium, that could have almost any shape: rectangle, rhombus, irregular triangle, etc.

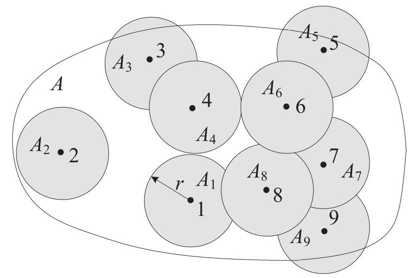



Fig. 1. An illustration of the random network deployment.

Motivated by the abovementioned limitations, instead of modeling the positions of nodes on the landscape explicitly, in this letter, we propose to use the randomly positioned interference areas (sets) created by nodes in the network. Using the tools of integral geometry, specifically, relying on the notion of kinematic measure of the group of motions of interference sets in a plane, we propose a simple methodology to approximate the distribution and moments of interference.

We introduce our methodology for the case of fixed number, $N$, interfering nodes distributed in a convex set with perimeter $L$ and area $S$ – one of the scenarios, where the stochastic geometry fails to provide an analytical Laplace transform. We validate the accuracy of our solution by comparing the numerical results it provides with the ones obtained via computer simulations. We also show the feasibility of the proposed approach to study the scenarios with a fixed number of interferers uniformly distributed in a rectangle, right triangle and half-circle. Finally, we briefly discuss the attractive extensions of the proposed framework that require further investigations.

## II. The Proposed Framework

### A. Network Model

The network deployment is shown in Fig. 1. The set $A$ denotes the area on interest. The only requirement we impose on $A$ is convexity. There are $N$ nodes randomly deployed in the area according to some arbitrary two-dimensional probability density function (pdf) $f(x, y)$. We are interested in pdf and moments of interference at a uniformly distributed point $P \in A$.

Manuscript received August 12, 2016; accepted September 11, 2016. Date of publication September 16, 2016; date of current version December 8, 2016. This work was supported by Academy of Finland FiDiPro program Nanocommunication Networks, 2012–2016. Dr. J. M. Jornet acknowledges the support of AFRL, under AFRL Grant No. FA8750-15-1-0050. V. Petrov acknowledges the support by Nokia Foundation. The associate editor coordinating the review of this letter and approving it for publication was J. Lee.

V. Petrov, D. Moltchanov, and Y. Koucheryavy are with the Department of Electronics and Communications Engineering, Tampere University of Technology, 33720 Tampere, Finland (e-mail: dmitri.moltchanov@tut.fi; yk@cs.tut.fi).

P. Kustarev is with the Department of Embedded Computing Systems, ITMO University, Saint Petersburg 197101, Russia (e-mail: kustarev@lmt.ifmo.ru).

J. M. Jornet is with the Department of Electrical Engineering, University at Buffalo, The State University of New York, Buffalo, NY 14260 USA (e-mail: jmjornet@buffalo.edu).

Digital Object Identifier 10.1109/LCOMM.2016.2610435

The stochastic geometry approach to the abovementioned problem is to characterize distributions of the distances $d_i$, $i = 1, 2, \dots, N$, between point $P$ and interferers in $A$ and then use a propagation model to represent distributions of individual contributions to the interference at $P$, [2], [4]. The distribution of the aggregated interference is obtained in the transform domain. This approach is feasible when the interferers are uniformly distributed in $A$ ($f(x, y) = 1/S_A$, where $S_A$ is the area of $A$), $A$ has some form of symmetry, while $P$ is the geometrical center of $A$. When any of these assumptions are relaxed, the distributions of $d_i$, $i = 1, 2, \dots, N$ are not available.

To tackle the formulated problem we propose to replace the first step in the analysis explained above. Particularly, we use the integral geometry to obtain the distribution of individual contributions at $P$ using *interference sets* introduced below. The rest of the analysis is intact, i.e., we obtain the distribution of the aggregated interference in the transform form.

### B. Interference Sets

The first step in our analysis is to characterize the area and perimeter of the interference set $A_1(x)$ created by a single interferer such that the interference in this area is at least $x$. Under the assumption of omnidirectional antennas, this set is of circular shape with area $\pi r^2$ and perimeter $2\pi r$. Let $r(x)$ be the radius of the interference set corresponding to a certain emitted power $P_{Tx}$ such that $x = P_{Rx}$.

Fixing $r$, the received power at $r$ is

$$P_{Rx}(r) = P_{Tx} L(r), \tag{1}$$

where $P_{Tx}$ is the emitted power and $L(r)$ is the pathloss.

We use the path loss model in the following form

$$L(r) = C r^{-\gamma}, \tag{2}$$

where $C$ and $\gamma$ are propagation-dependent constants.

Substituting (2) into (1) and inverting the resulting function, we get the radius of the interference zone $r$ with at least $x = P_{Rx}$ interference power inside as

$$r_{A_1}(x) = (x/CP_{Tx})^{-1/\gamma}, \tag{3}$$

where $x$ is the power at the circumference.

The perimeter and area of interference set $A_1(x)$ are now

$$L_{A_1}(x) = 2\pi(x/CP_{Tx})^{-1/\gamma}, \quad S_{A_1}(x) = \pi(x/CP_{Tx})^{-2/\gamma}. \tag{4}$$

### C. Interference Analysis With Integral Geometry

A motion of a point $P \in \Re^2$ or a set $A \subset \Re^2$ is defined as transformation of the coordinates by horizontal, $x$, vertical, $y$, and rotational, $\phi$, displacements. A group of motions, $M$, is the set of transformations in $\Re^2$, $(x, y, \phi)$. To quantify a group of motions in $\Re^2$ we need to define a measure for the set of transformations of $A$ described by $M$. This measure should be invariant under translation and rotation of the coordinate axis. The following definition of the element of the set of motions $M$ of a set $A$ (up to a constant factor) induces the only translationally and rotationally invariant measure.

*Definition 1 (Kinematic Density [5]): Let $M$ denote the group of motions of a set A in the plane. The kinematic density $dA$ for the group of motions $M$ in the plane for the set $A$ is*

$$dA = dx \wedge dy \wedge d\phi, \tag{5}$$

*where $\wedge$ is exterior product [6], $x$, $y$ are coordinates, $\phi$ is the rotation angle of $A$ with respect to $x$-axis.*

Now we introduce the kinematic measure.

*Definition 2 (Kinematic Measure [5]): The kinematic measure $m$ of a set of group motions $M$ on the plane is defined as the integral of the kinematic density $dA$ over $M$, that is,*

$$m_A = \int_M dA = \int_M dx \wedge dy \wedge d\phi. \tag{6}$$

The ratio of the measure of a group of motions $W$ and the measure of a group of motions $M$ in the plane, where $W \subseteq M$, provides the probability for this group of motions to happen, i.e., $p(W) = m(W)/m(M)$ enabling solution of various geometrical probability problems [5].

Consider now a single interferer in the area of interest. We are interested in the probability $p$ that an arriving node (point $P$) will fall into the interference area of the single interfering node, $r_{A_1}(x)$. Using conditional probability we write

$$p = Pr\{P \in A \cap A_1\}/Pr\{A \cap A_1 \neq 0\}. \tag{7}$$

Using the kinematic measure, we get [5]

$$\begin{aligned} Pr\{P \in A \cap A_1\} &= m(A : P \in A \cap A_1), \\ Pr\{A \cap A_1 \neq 0\} &= m(A : A \cap A \neq 0), \end{aligned} \tag{8}$$

where the first expression is the kinematic measure of the set of motions of $A$ such that $P \in A$, while the second one provides the measure of all motions of $A$, for which the intersection between $A$ and $A_1$ is non-zero.

Following [5], the first measure is

$$m(P \in A \cap A_1\}) = \int_{P \in A_1} f(x, y) dx \wedge dy \wedge d\phi, \tag{9}$$

where $f(x, y)$ is the density of interferers positions in $A$. Note that in (9) we initially need to integrate the kinematic density of $A_1$ over all motions of $A_1$ such that $P \in A \cap A_1$. However, since is assumed to fall in $A_0$ and $A_0$ is fixed we only need to integrate over all possible locations of $P$ such that $P \in A_1$.

The measure of all motions of $A$, such that $A \cap A_1$, is [5]

$$m(A \cap A_1 \neq 0) = \int_{A \cap A_1 \neq 0} f(x, y) dx \wedge dy \wedge d\phi. \tag{10}$$

According to (7), the probability that an arriving node will experience an interference greater than $x$, i.e., the complementary cumulative distribution function (CCDF), is obtained by substituting (9) and (10) into (7),

$$Pr\{I > x\} = \frac{\int_{P \in A \cap A_1} f(x, y) dx \wedge dy \wedge d\phi}{\int_{A \cap A_1 \neq 0} f(x, y) dx \wedge dy \wedge d\phi}, \tag{11}$$

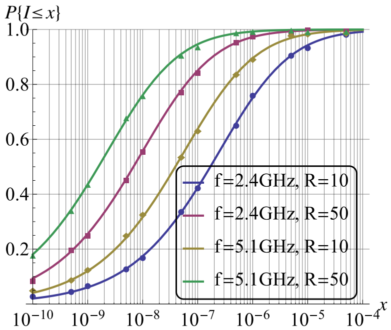
that can be computed for a particular forms of $A$, $A_1$, $f(x, y)$.

## III. Illustrative Examples

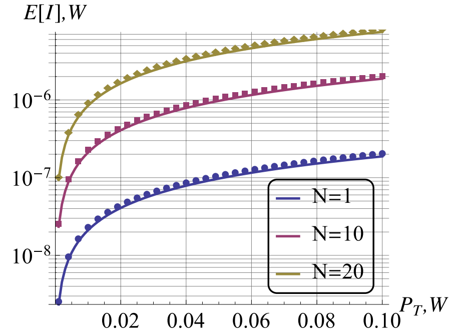
### A. N Interferers Uniformly Distributed in a Circle

As a first example consider the case of $N$ uniformly distributed interferers in a convex area. Let us first concentrate on a single interferer. Following [5], the first measure is

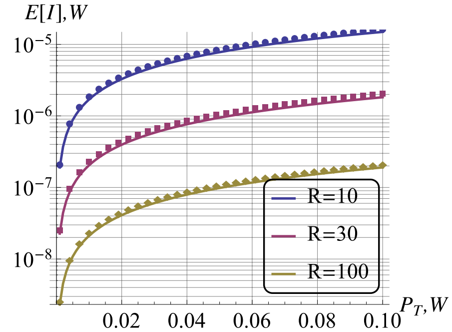
$$\begin{aligned} m(P \in A \cap A_1\}) &= \int_{P \in A_1} dx \wedge dy \wedge d\phi = \\ &= \int_{P \in A_1} dx \wedge dy \int_0^{2\pi} d\phi = 2\pi S_{A_1}(x). \end{aligned} \tag{12}$$



(a) CDF of interference from a single node



(b) Mean aggregated interference, $R = 30$



(c) Mean aggregated interference, $N = 10$

Fig. 2. The interference metrics associated with $N$ interferers scenario.

The measure of motions of $A$ such that $A \cap A_1 \neq 0$ is [5]

$$m(A \cap A_1 \neq 0) = \int_{A \cap A_1 \neq 0} dx \wedge dy \wedge d\phi = \\ = 2\pi [S_A + S_{A_1}(x)] + L_A L_{A_1}(x), \quad (13)$$

where $S_A$ and $L_A$ are the area and the perimeter of $A$.

CCDF of interference is then

$$Pr\{I > x\} = \frac{2\pi S_{A_1}(x)}{2\pi [S_A + S_{A_1}(x)] + L_A L_{A_1}(x)}. \quad (14)$$

For the circularly shaped zone $A$ of radius $R$ we have

$$Pr\{I > x\} = \left(\frac{r_{A_1}(x)}{r_{A_1}(x) + R}\right)^2. \quad (15)$$

Substituting $S_{A_1}(x)$, $L_{A_1}(x)$ from (4) into (15) yields

$$Pr\{I > y\} = (Ry^{1/\gamma} + 1)^{-2}, \; y = x/CP_{Tx}. \quad (16)$$

Differentiating $1 - Pr\{I > x\}$ we obtain the pdf as

$$f_I(x) = \frac{2Ry^{1/\gamma}}{\gamma y C P_{Tx} (Ry^{1/\gamma} + 1)^3}. \quad (17)$$

The interference from $N$ nodes is obtained by switching to the Laplace transform domain and calculating $[L_I(s)]^N$. The general form of the transform of (17) is rather complex. A special case with $P_{Tx} = 1$, $A = 1$, $\gamma = 2$ is given in (18), as shown at the bottom of this page, where Chi(·) and Shi(·), are the hyperbolic cosine and sine integrals, respectively, Erf(·) is the error function. The $N$th power of this transform in (18) is differentiable and can be used to estimate the moments of the aggregated interference from $N$ nodes. One can use numerical inverse of $[L_I(s)]^N$ using, e.g., Talbot method [7], to get the density of the aggregated interference.

The critical step in the proposed framework is the derivation of the pdf of interference from a single node as the rest of the analysis follows well-accepted steps. Fig. 2a shows the CDFs of the interference obtained using (15) and simulations for 1W of emitted power. The modeling results are illustrated by lines, while the simulation ones are marked by points. The parameters of the scenario are: (i) circular interference zone of radius $R$, (ii) free-space propagation model and (iii) uniform distribution of interfering nodes. As one may observe, the proposed model follows simulation results precisely. This illustration allows to conclude that the derivation of the interference according to the proposed model is accurate. Fig. 2b and 2c show the mean aggregated interference (in logarithmic scale) for different number of interferers, different radii of the interference zone and operational frequency 2.4GHz obtained using the proposed approach and simulation of the considered scenario. As can be seen, the proposed model slightly underestimates the actual interference.

Let us demonstrate the typical stochastic geometry approach to this problem. The distance between two points uniformly distributed in the circle of radius $R$ is given by [8]

$$f_X(x) = \frac{4x}{\pi R^2}\left(\cos^{-1}\left(\frac{x}{2R}\right) - \frac{x}{2R}\sqrt{1 - \frac{x^2}{4R^2}}\right). \quad (19)$$

Transforming (19) using the propagation model $P_{Rx}(r) = P_{Tx} C r^{-\gamma}$, letting $y = x/CP_{Tx}$, and omitting intermediate calculations we get the pdf of interference from a single source,

$$f_I(y) = \frac{4Ry^{1/\gamma} \sec^{-1}\left(2Ry^{1/\gamma}\right) - \sqrt{4 - y^{-2/\gamma} R^{-2}}}{\pi \gamma R^3 C P_{Tx} y^{(3/\gamma + 1)}}. \quad (20)$$

The pdf in (20) does not have analytical representation in the Laplace domain even for special case $P_{Tx} = 1$, $A = 1$, $\gamma = 2$ preventing derivation of the the aggregated interference pdf.

### B. Indoor Communications With Complex Geometries

A good example of the application area, where the integral geometry shows its versatility, is indoor communications in

$$L_I(s) = R^{-4} s e^{-sR^{-2}} \left(R^2 - 2s\right)\left[\mathrm{Chi}\left(sR^{-2}\right) - \pi \,\mathrm{Erf}\left(sR^{-1}\right) + \mathrm{Shi}\left(sR^{-2}\right) - \log\left(R^{-2}\right) - 2\log(R)\right] \\ + R\left(R^3 + 2Rs - 2\sqrt{\pi} s^{3/2}\right). \quad (18)$$

complex geometries. As we have seen, even for a fully symmetric case of a circle the stochastic geometry framework does not allow to solve the system for the aggregated interference. Considering triangular indoor area as an example, the distribution of the distance between two points uniformly distributed over the area is only available for right triangles only [9]. It is easy to check observing the structure of the density in [9] that even in this special case there is no solution for the Laplace transform of the density of interference from a single source. Similar conclusions hold for other geometries, e.g., rectangles, half-circles, see [10].

Using (14), the densities of interference from a single source for rectangle, with sides $a$ and $b$, right triangle with legs $a$ and $b$, and half-circle of radius $R$ are respectively given by

$$f_I^{[R]}(y) = \frac{2\pi y^{1/\gamma}\left(aby^{1/\gamma}+a+b\right)}{\gamma yCP\left[y^{1/\gamma}\left[aby^{1/\gamma}+2(a+b)\right]+\pi\right]^2},$$
$$f_I^{[T]}(y) = \frac{4\pi y^{1/\gamma}\left[c+aby^{1/\gamma}+a+b\right]}{\gamma yCP\left[y^{1/\gamma}\left(2\left(c+b\right)+a\left[by^{1/\gamma}+2\right]\right)+2\pi\right]^2},$$
$$f_I^{[H]}(y) = \frac{4Ry^{1/\gamma}\left(Ry^{1/\gamma}+1\right)}{\gamma yCP\left[Ry^{1/\gamma}\left(Ry^{1/\gamma}+2\right)+2\right]^2}, \quad (21)$$

where $y = x/CP$. As one may check, in special cases, these densities have analytical Laplace transform representation facilitating similar analysis of the aggregated interference, as presented in the previous subsection.

## IV. Possible Extensions

*1) Clusterization of Interferers:* Let the positions of the interfering nodes follow a general distribution with pdf $f(x, y)$ in the area of interest. In this case, the kinematic measures are

$$m(P \in A \cap A_1) = \int_{P\in A_1} f(x, y)dx \wedge dy \wedge d\phi,$$
$$m(A \cap A_1 \neq 0) = \int_{A_0\cap A\neq 0} f(x, y)dx \wedge dy \wedge d\phi. \quad (22)$$

where the integrals may or may not have closed-form solutions for specific $f(x, y)$. Nevertheless, they can always be computed numerically. This extension is of special interest for modeling clusterization of users.

*2) Random Number of Interferers:* The proposed approach can be also extended to take into account random number of interferers in the area $A$. As an example, let the interferers be distributed over the landscape according to PPP with intensity $\lambda$ and let $A$ be a convex area of interest. Letting $G(z)$ be the probability generating function of the Poisson distribution with mean $\lambda S_A$ we get the transform of the interference distribution in the form $G(L_I(s))$.

*3) Directional Antennas:* The directivity of interfering nodes' antennas can be captured by defining an appropriate model for the antenna radiation pattern and then applying the abovementioned analysis. The cone antenna model [11] has triangular radiation pattern and can be used as it is.

*4) Fading:* The fading phenomena (e.g., Rayleigh, Rician or Nakagami fading) can be taken into account by appropriately transforming the distribution of interference from a single node using conventional random variable transformation techniques and then following the analysis described above. Only numerical analysis is likely feasible for this case.

*5) 3D Case:* The principal difference is that now sets $A$ and $A_i$ are three-dimensional ones. While the general expression for the interference provided in (11) remains intact, the derivations of measures in (12) and (13) have to be updated accordingly. An interested reader is referred to [5, Ch. 15] for further details. The ability of the proposed approach to tackle interference analysis in three-dimensional space is of special interest as the deployment regions may no longer have symmetric configurations (e.g., as a result of ground effects) complicating the use of the stochastic geometry approach.

## V. Conclusions

We proposed a new framework for interference analysis of wireless systems with arbitrary node deployments in arbitrary areas. The principle difference compared to the conventional stochastic geometry approach is that the complex problem of finding the individual contributions of the interfering nodes is replaced by less involved problem of characterizing overlapping between interference sets. The latter is facilitated by application of the integral geometry.

The results provided by the developed framework are approximate in nature. The reason is that the interference zone geometry is not confined for interference sets and gets larger as the argument of the CDF increases. Still, the interference zone for a new arriving node is well defined and the approximation is accurate, as confirmed in Fig. 2. The reason is that most of the interference comes from the nearby nodes. For realistic dimensions of the interference zone and density of interferers the impact of those nodes that are far away is negligible.

The developed framework allows for a number of extensions including (i) random and fixed number of interferers, (ii) non-uniform distribution of interferers, (iii) directional transmit and receive antenna patterns, and (iv) small-scale fading. Combinations of the abovementioned extensions allow to address a number of important cases of wireless networks operation.

## References

[1] Y. Li, F. Baccelli, H. Dhillon, and J. Andrews, "Statistical modeling and probabilistic analysis of cellular networks with determinantal point processes," *IEEE Trans. Commun.*, vol. 63, no. 9, pp. 3405–3422, Sep. 2015.
[2] M. Haenggi, *Stochastic Geometry for Wireless Networks*. Cambridge, U.K.: Cambridge Univ. Press, 2012.
[3] S. Zuyev and F. Baccelli, "Stochastic geometry models of mobile communication networks," in *Proc. Frontiers Queueing Models Appl. Sci. Eng.*, 1997, pp. 227–243.
[4] V. Petrov, D. Moltchanov, and Y. Koucheryavy, "Interference and SINR in dense terahertz networks," in *Proc. IEEE Veh. Technol. Conf.*, Sep. 2015, pp. 1-5.
[5] L. Santalo, *Integral Geometry*. Reading, MA, USA: Addison-Wesley, 1976.
[6] H. Flanders, *Differential Forms*. Englewood Cliffs, NJ, USA: Prentice-Hall, 1967.
[7] A. Cohen, *Numerical Methods for Laplace Transform Inversion* (Numerical Methods and Algorithms). New York, NY, USA: Springer, 2007.
[8] D. Moltchanov, "Distance distributions in random networks," *Ad Hoc Netw.*, vol. 10, pp. 1146–1166, Aug. 2012.
[9] U. Bäsel. (Sep. 2012). "The distribution function of the distance between two random points in a right-angled triangle." [Online]. Available: https://arxiv.org/abs/1208.6228
[10] A. Mathai, *An Introduction to Geometrical Probability: Distributional Aspects with Applications*, 3rd ed. Boca Raton, FL, USA: CRC Press, 1999.
[11] S. Singh, R. Mudumbai, and U. Madhow, "Interference analysis for highly directional 60-GHz mesh networks," *IEEE/ACM Trans. Netw.*, vol. 19, no. 5, pp. 1513–1527, Oct. 2011.